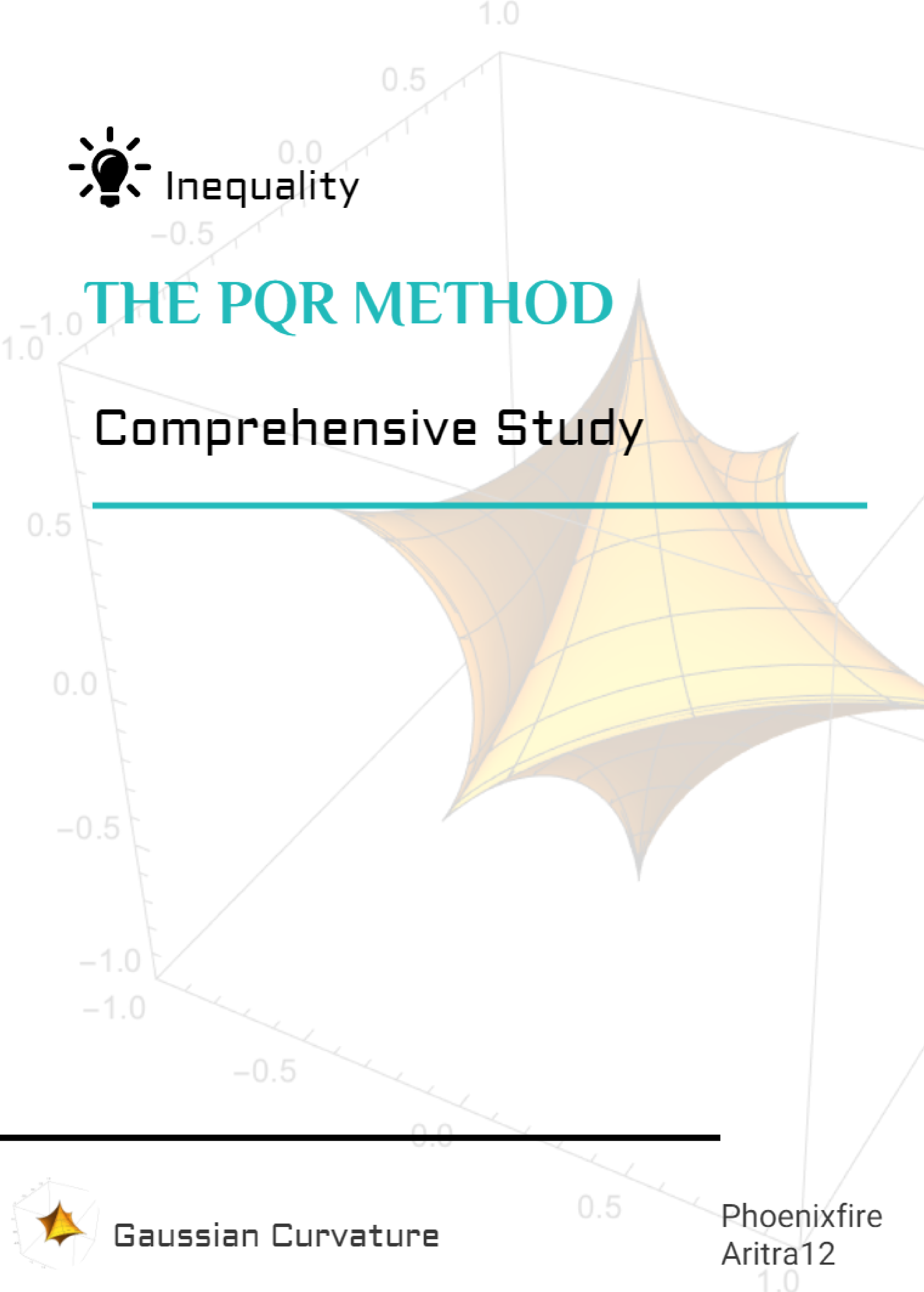
Inequality

THE PQR METHOD

Comprehensive Study



Gaussian Curvature

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Chapter 1

The pqr Method

1.1 Introduction

If there is a 50-50 chance that something can go wrong, then 9 times out of ten it will. - Paul Harvey

1.1.1 The Basics of the Basics

The pqr/uvw method is a useful technique for proving inequalities involving symmetric polynomials in three real non-negative variables. This method is highly related with Schur's inequality. These types of problems are common in math Olympiads. The basic idea is to introduce a specific change of variables that simplifies the original inequality. Every symmetric polynomial in a,b,c (or x,y,z) can be written as a polynomial in p,q,r or u,v^2,w^3.

The method is mostly used to prove inequalities or used to show that the maximum or minimum value of an expression involving non-negative real numbers a,b,c is 16 Code check failed attained when two of the variables are equal or one of the variables is zero. This requires some knowledge about the method so by all means continue reading. Note that this method also has a geometric interpretation.

Using the method to find the minimum/maximum in the general way is not particularly powerful. The real power of the method comes from a general result known as Tejs' theorem, which is stated and proved below (yeah spoilers). Tejs' theorem shows that under certain circumstances (which you will come to know later), the maximum or minimum of a symmetric expression in three non-negative real variables occurs when two of the variables are equal, or one of the variables is 0 (See the relation with Schur's). In fact, the theorem can be used in more complicated cases too (see the section on warning) in which more types of triples (a,b,c) must also be checked.

1.1.2 How to Use This Handout

This handout is by no means the best handout for the topic, and also this handout is just to get you started. This handout has both theory, and problems. I first show you symmetric polynomials in two variables and then in three variables.

I have tried not to give a greater importance to any one method over the other switching between both methods but the solutions are mostly using pqr. Where I use which method has been explicitly mentioned.

I also switch between a,b,c and x,y,z due to polynomial reasons, this too has been explicitly mentioned. I mostly use pqr or uvw in this handout and generally use a,b,c for both of them. The switch has also been included in the problems just to make the reader comfortable.

I suggest that you read all sections of theory and then try to solve the problems. Thus making you more comfortable with this technique and making you faster at problem solving.

1.1.3 My first Handout

This is my first time writing handouts like these, so please remember that I am a human and I may make mistakes. There may be typos, grammatical errors, but I will try to be as clear as possible.

I apologize if there are any typos, mistakes, confusing parts. This handout may be loaded with weird, inconsistent capitalization and grammar mistakes and indenting. If you find any, please mention them in this thread or contact me on AoPS.

I am sorry if you didn't like this and please give me your honest feedback! I also apologize if my formatting was annoying. This is my first time with Overleaf so I did not get everything I wanted. I may make more handouts like these in the future so your feedback is appreciated (I currently have one in the making on Inequalities in general and a few techniques).

Also sorry for making it very very long.

1.1.4 History

As far as I know, the pqr Method was originally from Vietnam, where it was known as abc Method (i.e. abstract-concreteness method).

This method has been popularized by Michael Rozenberg (arqady). The abc Method became a uvw Method, and was modified by M. Rozenberg.

1.1.5 A Word of Thanks

First of all I would like to thank Aritra12 for modifying this handout. I would also like to thank him for his suggestions. Next I would like to thank M. Rozenberg and others who made handouts for this method. And of course all of those people who were involved in the making and creating of this handout.

1.2 The Basic Theory

1.2.1 Without Loss of Generality

This is a phrase commonly used in the world of inequalities. However, it must be used with care.

Example 1.2.1. Suppose were trying to prove that $a^2b+b^2c+c^2a-3\geq 0$ when $abc=1$ and $a,b,c>0$. Why would it be a very bad idea to start the solution with the phrase Without loss of generality, $a\geq b\geq c$?

Suppose we assume $a\geq b\geq c>0$ and we have $\frac{a}{b}+\frac{b}{c}+\frac{c}{a}-3=G(a,b,c)$. Then observe that $G(a,b,c)\geq 3$ is a cyclic inequality, but not symmetric. Therefore, we cannot assume any pairwise order between a,b,c, but we can assume that one of them is the minimum or maximum for all of them. Also, it is not difficult to prove that, if $a\geq b\geq c$, then $G(a,b,c)\leq G(a,c,b)$. What happens when $abc=1$ and you multiply the variable terms of $G(a,b,c)$ by abc?

1.2.2 Symmetry

We call expression $f(a_1,a_2,,a_n)\geq 0$ symmetric, if its value does not change after swapping any two variables (i.e. $f(a_1,a_2,,a_n)=f(a_2,a_3,,a_n,a_1)=\dots$). Because of the symmetry, we can rearrange the order of variables (that means we can choose an arbitrary order). Because of the symmetry, we can estimate a mixed expression by smaller expressions of one-variable. The fundamental intuition is being able to decide which symmetric polynomials of a given degree are bigger. For example, for degree 3, the polynomial $a^3+b^3+c^3$ is biggest and abc is the smallest. Roughly, the more mixed polynomials are the smaller

1.2.3 Two variables

> **Theorem 1.2.1** — p,q by convenience: $p=a+b,q=ab$. Obviously, a,b are roots of the quadratic equation $x^2-px+q=0$ (two of a,b can be equal or complex).

If p and q are real, then either a and b are both real or b is the complex conjugate of a. Further $(a-b)$ is either real or pure imaginary.

Exercises

1. Which conditions (particularly, inequalities) should satisfy p and q for a and b to be real?

2. Prove that only if p and q are non-negative real numbers which satisfy conditions from the previous problem then a and b are real and non-negative.

1.3 The Method

This method is a powerful instrument which can be used for proving inequalities of varying difficulty which cannot be proved with other methods and techniques. It should be noted at this point that this method works for all symmetric inequalities.

Theorem 1.3.1 — If f_3 is a polynomial, then it can be rewritten in terms of p,q,r by convenience: $p=a+b+c, q=ab+bc+ca, r=abc$. Obviously, a,b,c are roots of the cubic equation $x^3-px^2+qx-r=0$ (two of a,b,c can be equal or complex).
The theorem can be restated as: Prove that any symmetric polynomial in a, b, c can be expressed as a polynomial in p, q, r.

To prove this we need the following lemma:

Lemma 1.3.2 — Let $s_k=a^k+b^k+c^k$ for any non-negative integer number k. It is possible to express x^k for $k>3$ in terms of p,q,r,x_{k-1},x_{k-2} and x_{k-3}.

Proof.

$$ps_{k-1}-qs_{k-2}+rs_{k-3}$$
$$=(a+b+c)(a^{k-1}+b^{k-1}+c^{k-1})-(ab+bc+ca)(a^{k-2}+b^{k-2}+c^{k-2})+abc(a^{k-3}+b^{k-3}+c^{k-3})$$

$$=(s_k+ab^{k-1}+ac^{k-1}+ba^{k-1}+bc^{k-1}+ca^{k-1}+cb^{k-1})$$
$$-(ab^{k-1}+ac^{k-1}+ba^{k-1}+bc^{k-1}+ca^{k-1}+cb^{k-1}+abc^{k-2}+ab^{k-2}c+a^{k-2}bc)$$
$$+(abc^{k-2}+ab^{k-2}c+a^{k-2}bc)=s_k$$

Proof of theorem. Let G(a, b, c) be a given polynomial. $G=G_1+G_2+G_3$, where all monomials in G_i contain $i$variables, $i=1,2,3$. From the previous theorem it follows that G_1 can be expressed as a polynomial in p,q,r. Since an equality

$$s_ks_l-s_{k+l}=a^kb^l+a^kc^l+b^ka^l+b^kc^l+c^ka^l+c^kb^l$$

holds, it follows that G_2 can be expressed as a polynomial in p,q,r. Finally, for a sum $a^kb^lc^m+$ something in G_3 we factorize $(abc)^n$ (where $n=min(k,l,m)$) and reduce our problem to the previous cases.

Theorem 1.3.3 — Given $p,q,r\in\mathbf{R}$; $a,b,c\in\mathbf{R}$ such that $p=a+b+c, q=ab+bc+ca, r=abc$ if and only if: $p^2\geq 3q$ and

$$r\in\left[\frac{9pq-2p^3-2\sqrt{(p^2-3q)^3}}{27},\frac{9pq-2p^3+2\sqrt{(p^2-3q)^3}}{27}\right]$$

Or $r_{min}(p,q)=\frac{(p-2\sqrt{p^2-3q})(p+\sqrt{p^2-3q})^2}{27}$ and $r_{max}(p,q)=\frac{(p-\sqrt{p^2-3q})^2(p+2\sqrt{p^2-3q})}{27}$

Proof: Try to prove it by yourself. You will require:

> **Lemma 1.3.4** — Assume that p,q, and r are real numbers. Prove that if a,b,c are real, then
> $$(a-b)(b-c)(c-a)$$
> is real, otherwise it is pure imaginary.

> **Lemma 1.3.5** — Prove that
> $$T(p,q,r) = (a-b)^2(b-c)^2(c-a)^2 = -4p^3r + p^2q^2 + 18pqr - 4q^3 - 27r^2$$

When making a change of variables in general, it is often quite important to understand its inverse. In this particular case, passing from the variables a,b,c to p,q,r is relatively easy to understand.
But suppose we are given (non-negative) values of p,q,r, to get the corresponding a,b,c is a more important and difficult process, and some values of p,q,r may not even correspond to real values of a,b,c.

> **Theorem 1.3.6** — If $p,q,r \geq 0$ and $T(p,q,r) \geq 0$ then a,b,c are real and non-negative Thus it can be said that if a,b,c are non-negative and $p = a+b+c, q = ab+bc+ca, r = abc$, then
> $$\frac{p}{3} \geq \sqrt{\frac{q}{3}} \geq \sqrt[3]{r} \leftrightarrow p^6 \geq 27q^3 \geq (27r)^2$$
> with equality holding if and only if $a = b = c$ or if two of a,b,c are 0.

The theorem gives conditions that are necessary and sufficient for values of p,q,r to correspond to real values of a,b,c. See the connection with Schur's yet?
Hint: Proceed by contradiction - If a,b,c are not all non-negative, what do you get? Without loss of generality assume $a \leq 0$. For the converse - Cubic polynomial and discriminant.

Example 1.3.1. Let the positive real numbers x,y,z satisfy $x+y+z+9xyz = 4(xy+yz+zx)$. Show that $x+y+z \geq 1$.

Let $p = x+y+z, q = xy+yz+zx, r = xyz$, thus we have to prove $p \geq 1$. By Schur's inequality for $t = 1$
$$p^3 + 9r \geq 4pq = p^2 + 9pr$$
$$\Rightarrow (p-1)[p^2 - 9r] \geq 0$$
If $x+y+z < 1$, then $p^2 > p^3 \geq 27r > 9r$ contradiction. Thus $p \geq 1$.

1.4 The uvw Method

The method uses the substitution $3u = a+b+c, 3v^2 = ab+bc+ca, w^3 = abc$ instead of $p = a+b+c, q = ab+bc+ca, r = abc$. Most of the times. But some times $u = a+b+c, v^2 = ab+bc+ca, w^3 = abc$ is also used, thus making it similar to the *pqr* method. Over here $3u = p, 3v^2 = q, w^3 = r$, and note that $3v^2$ can be negative for obvious reasons.

If a,b,c are non-negative and u,v,w are the substitutions in the *uvw* method, then $u \geq v \geq w$, with equality holding if and only if $a = b = c$ or (for $v \geq w$) if two of a,b,c are 0.

1.5 The abc Method

The *abc* method uses the substitution $a = x+y+z, b = xy+yz+zx, c = xyz$, similar to the *pqr*. This is method even though the same is relatively less well known. And thus for further purposes we will stick with either the *pqr* or *uvw* method. (Sorry)

Exercise:

3.1. Let a,b,c be positive reals such that $a+b \geq c$, $b+c \geq a$ and $c+a \geq b$. Prove that

$$2a^2(b+c)+2b^2(c+a)+2c^2(a+b) \geq a^3+b^3+c^3+9abc$$

Hint: Use the incircle substitution including the degenerate case

3.2.1. Express s^k for $1 \leq k \leq 6$ in terms of *pqr*. You can check your expressions in Shortcuts.
Hint: Express $(a+b+c), (a+b+c)^2, (a+b+c)^3$, then can you continue?

3.2.2. Let a,b,c be real numbers such that $a+b+c = 9$, $ab+bc+ca = 24$. Prove that $16 \geq abc \geq 20$. Prove moreover that for any $r \in [16,20]$ there exist real numbers a,b,c such that $a+b+c = 9$, $ab+bc+ca = 24$, $abc = r$.
Hint: Substitute the values in $T(p,q,r)$.

3. Suppose a,b,c are positive real numbers such that $abc = 1$ and

$$\frac{1}{a}+\frac{1}{b}+\frac{1}{c} = 1 = \frac{1}{ab}+\frac{1}{bc}+\frac{1}{ca}$$

Find the minimum value of $(a+1)(b+1)(c+1)$.
Hint: Clear denominators.

Things to note

1.6 Symmetric polynomials

All three (i.e. pqr, uvw, abc) methods are best for symmetric polynomials of low degree and may still require quite a lot of computation after applying Tejs' theorem (discussed later). A solution with any one method implies a solution with the others.

When not to use

Also, they are not always the best choice when dealing with square roots, inequalities of a very high degree of simply non-symmetric or more than 4 variable inequalities. You probably want to consider using another technique if this happens, but a solution is possible.

1.7 Qualitative estimations

It can be really tedious to write everything in terms of u, v^2, w^3.
Because of this it can be really useful to know some "qualitative estimations". We already have some bounds on w^3, but they are not always (that is, almost never) nice. The square root tends to complicate things, so there is indeed a better way, than to use the bounds always!

If you havent noticed: Many inequalities have equality when $a = b = c$. Some have when $a = b, c = 0$, and some again for $a = b = kc$ for some k. There is rarely equality for instance when $a = 3, b = 2, c = 1$, although it happens.

There is a perfectly good reason for this: When we fix two of u, v^2, w^3, then the third assumes its maximum if and only if two of a, b, c is equal! (You will get to know why it is such in Tejs' Theorem, go on read ahead)

Consequences

Here are some very handy and helpful tools (but be aware that you do have to prove them before using them unlike some theorems/properties) to ease your life while using this method.

If a,b,c are non-negative real numbers and we denote $p=a+b+c, q=ab+bc+ca, r=abc$, then:

$$pq \geq 9r$$
$$p^2 \geq 3q$$
$$q^2 \geq 3pr$$
$$p^2q+3pr \geq 4q^2$$
$$pq^2 \geq 2p^2r+3qr$$
$$p^2q^2+12r^2 \geq 4p^3r+pqr$$
$$p^3 \geq 27r$$
$$q^3 \geq 27r^2$$
$$p^3r \geq q^3$$
$$p^3+9r \geq 4pq$$
$$2p^3+9r \geq 7pq$$
$$2p^3+27r \geq 9pq$$
$$2p^3+9r^2 \geq 7pqr$$
$$q^3+9r^2 \geq 4pqr$$
$$2q^3+27r^2 \geq 9pqr$$
$$p^4+3q^2 \geq 4p^2q$$
$$p^4+4q^2+6pr \geq 5p^2q$$
$$4p^5q+44p^2qr+17pq^3 \geq 4p^4r+20p^3q^2+25q^2r+24pr^2$$

Equality occurs if and only if $a=b=c$.

Exercise:

Prove the above inequalities.

1.8 Tejs' Theorem

1.8.1 p-lemma

Fix some values $q = q_0$ and $r = r_0 > 0$ such that there exists at least one value of p for which the triple (p, q_0, r_0) is acceptable. Then p has a global maximum and minimum. p assumes maximum and minimum only when two of a, b, c are equal.
If $r_0 = 0$, then $p \in [2\sqrt{q_0}, \infty]$, i.e. p is unbounded.

1.8.2 q-lemma

Fix some values $p = p_0$ and $r = r_0$ such that there exists at least one value of q for which the triple (p_0, q, r_0) is acceptable. Then q has a global maximum and minimum. q assumes maximum and minimum only when two of a, b, c are equal.

1.8.3 r-lemma

Fix some values $p = p_0$ and $q = q_0$ such that there exists at least one value of r for which the triple (p_0, q_0, r) is acceptable. Then r has a global maximum and minimum. r assumes maximum only when two of a, b, c are equal, and minimum either when two of a, b, c are equal or when one of them are zero.

If $r = 0$, then one number of a, b, c equals 0. Proof (p-lemma) The condition that (p, q, r) is admissible is $T(p, q, r) \geq 0$, i.e.

$$-4p^3r + p^2q^2 + 18pqr - 4q^3 - 27r^2 \geq 0$$

With p, q fixed, this is a quadratic polynomial in r, say $f(r)$. The graph of f is a parabola pointing down. If $S_{p,q}$ is empty, there is nothing to prove, so assume it is nonempty.

Then we seek the set of non-negative r for which $f(r)$ is non-negative. The set of points x for which $f(x)$ is non-negative is an interval $[x_0, x_1]$.

Note that $x_1 \geq 0$ because otherwise there would be no non-negative r for which $f(r)$ was non-negative, so $S_{p,q}$ would be empty.

There are two cases.

Case 1.8.1. If $x_0 \geq 0$

The endpoints x_0 and x_1 are the minimum and maximum values of r, and they correspond to the values of r for which T is zero.
But $T(p,q,r) = 0$ if and only if $(a-b)^2(b-c)^2(c-a)^2 = 0$, which happens if and only if at least two of a,b,c are equal.
So the maximum and minimum occur at values where two variables are equal.

Case 1.8.2. If $x_0 \leq 0$

The minimum value of $S_{p,q}$ is $r = 0$. So $abc = 0$, so one of the variables is zero.
The maximum x_1 still corresponds to two of a,b,c being the same, by the same argument as the previous paragraph.

Example 1.8.1. Let $a,b,c \geq 0$ be real numbers and $a+b+c = ab+bc+ca$ then Prove that

$$(a+b+c)(a^2b^2+b^2c^2+c^2a^2) \leq (a^2+b^2+c^2)^2$$

Homogenize the inequality. Then $p = a+b+c = ab+bc+ca = q$, $r = abc$ Then we have to prove

$$\frac{p^2}{q}(q^2-2pr) \leq (p^2-2q)^2$$

Thus we need to show that

$$2p^3r - p^2q^2 + q(p^2-2q)^2 \geq 0$$

Fixing p and q. Note that the LHS of this is a linear function of r, hence the extrema are achieved when r achieves its extrema.
Firstly, wlog assume $b = c$. Then the LHS becomes

$$ab(a-b)^2(2a^2+3ab+4b^2)$$

which is obviously non-negative.
Now wlog assume $c = 0$, which means $p = a+b = ab = q$, $r = 0$, and the inequality reduces to

$$(a^2+b^2)^2 \geq (a+b)^2ab$$

which is obvious by AM-GM-QM.

Example 1.8.2. Let a,b,c be nonnegative real numbers such that $ab+bc+ca = 1$. Find the minimum value of

$$\frac{1}{a+b}+\frac{1}{b+c}+\frac{1}{c+a}$$

Setting $a = b = 1$ and $c = 0$ gives $\frac{5}{2}$. This is infact the minimum, and thus to prove that

$$(b+c)(c+a)+(a+b)(c+a)+(a+b)(b+c)-\frac{5}{2}(a+b)(b+c)(c+a) \geq 0$$

The left side is a symmetric polynomial in a,b,c of degree 3. In terms of u,v,w, it is $9u^2 + 3v^2 + \frac{5}{2}(w^3 - 9uv^2)$, which equals

$$\frac{5}{2}w^3 + 9u^2 - \frac{15}{2}u + 1$$

in our case (since $3v^2 = 1$).

For fixed u this is a linear polynomial in w^3, which will attain its minimum value either when $w^3 = 0$ or when w^3 is minimized subject to the admissibility constraint, which happens when two of the variables are equal or one of the variables is zero. So it is enough to look for the minimum value of this expression when two of the variables are equal or one of the variables is zero.

When one of the variables is zero, without loss of generality $a = 0$ and $bc = 1$. The expression becomes $9u^2 - \frac{15}{2}u + 1$, where $3u = b + c$, and $bc = 1$, so this factors as

$$\left(3u - \frac{1}{2}\right)(3u - 2) = \left(b + c - \frac{1}{2}\right)(b + c - 2)$$

But when $bc = 1$, clearly $b + c \geq 2$ (by AM-GM), so the expression is always nonnegative, and attains its minimum value of 0 when $b = c = 1$.

When two of the variables are equal, without loss of generality, $a = b$. Then rewriting in terms of b,c and using the factorization of $9u^2 - \frac{15}{2}u + 1$ gives the constraint as $b^2 + 2bc = 1$ and the expression as

$$\frac{5}{2}b^2c + \left(2b + c - \frac{1}{2}\right)(2b + c - 2) \geq 0$$

Making the substitution $c = \frac{1-b^2}{2b}$ gives

$$\frac{-5b^5 + 9b^4 - 10b^3 + 10b^2 - 5b + 1}{4b^2} = \frac{(1-b)(b^2+1)(5b^2 - 4b + 1)}{4b^2}$$

and each of the three factors on top (as well as the denominator) are always nonnegative for $b \in [0,1]$.

This interval is forced by the constraint $b^2 + 2bc = 1$. Thus we are done.

Exercise

1. Let a, b, c be non-negative real numbers such that $a+b+c=1$. Show that

$$1+12abc \geq 4(ab+bc+ca)$$

2. Can you similarly prove the q-lemma and r-lemma?
Hint: Cubic polynomial in q then consider then set of non-negative values of x for which $g(x)$ is non-negative, in this case $q=0$. Do the same for $r \neq 0$, but over here leading term is negative, do a similar argument, in this case $u=0$.

Important Corollaries

> **Corollary 1.8.3** — Every symmetric inequality of degree ≤ 5 in non-negative real variables a,b,c with a global minimum and/or maximum will attain this value at triples (a,b,c) with either two of the variables equal or one of the variables equal to zero. i.e. Only to be proved when $a = b$ and $a = 0$.

Proof. Fix p,q and consider the resulting polynomial in r; it can be written as the symmetric functions it is linear in r. Hence it is either increasing or decreasing.

So, its extrema occur when r is maximized or minimized. Thus, we only must check it when two of a,b,c are equal or when one of them is zero. Because of symmetry we can without loss of generality assume $a = b$ or $c = 0$.

> **Corollary 1.8.4** — Let f be a symmetric polynomial of degree ≤ 8 in non-negative real variables a,b,c. Write f as $Ar^2 + Br + C$, where A,B,C are functions of p,q. Then a global minimum and/or maximum of f, if it exists, will be attained at triples (a,b,c) with either two of the variables equal or one of the variables equal to zero, or in the places corresponding to solutions of $2Ar + B = 0$.

Proof. Fix p,q and notice that the resulting polynomial is at most quadratic in r. So, its extrema will occur either at points where r is maximized or minimized (i.e. at the endpoints of the domain) or when the quadratic polynomial is at a critical point.

Such critical points correspond to places where $2Ar + B = 0$, by elementary calculus. (The point is that this equation might be easier to analyse than the original one since it has lower degree.)

Warning

It is tempting to conclude that Tejs' theorem shows that any inequality involving a symmetric polynomial in three variables need only be checked when two of the variables are equal or one of the variables is zero. This is wrong.

Example 1.8.3. Let a,b,c be non-negative real numbers with $a+b+c = 12$ and $a^2 + b^2 + c^2 = 54$. Find the maximum value of $102abc - a^2b^2c^2$.

Hint: Using uvw, find a polynomial by applying Tej's Theorem say $x^3 + t_1x^2 + t_2x + t_3$ and then use the range of t_3.

Suppose we check only cases where two of the variables are equal or one variable is zero.

If $a = 0$ we get $b + c = 12$ and $b^2 + c^2 = 54$, so $b + c = 12$ and $bc = 45$, which is impossible by AM-GM. If $a = b$ we get $2a + c = 12$ and $2a^2 + c^2 = 54$, so $2a^2 + (12 - 2a)^2 = 54$, so $6a^2 - 48a + 90 = 0$, which factors as $6(a-3)(a-5) = 0$.

This leads to the two solutions $(3,3,6)(3,3,6)$ and $(5,5,2)$. In the first case, $abc = 54$, so $102abc - a^2b^2c^2 = 2592$, and in the second case, $abc = 50$, so $102abc - a^2b^2c^2 = 2600$. So we might erroneously conclude that the minimum is 2592 and the maximum is 2600, but this is incorrect.

In fact it is not hard to see that the possible values of abc given the constraints are in the closed interval $[50,54]$, by applying Tejs' theorem, or by looking at the graph of the cubic function $y = x^3 - 12x^2 + 45x - d$ for various values of d, and concluding that it crosses the x-axis three times if and only if $d \in [50,54]$.

However, the quantity $102abc - a^2b^2c^2$ is clearly maximized when $abc = 51$, which occurs at neither of the endpoints of the interval. The maximum value is therefore 2601, occurring when a,b,c are the three real roots of the polynomial $x^3 - 12x^2 + 45x - 51$.

This is the situation described in corollary (2) of Tejs' theorem; the maximum of the expression $102w^3 - w^6$ occurs either when two of the variables are equal or when the quadratic polynomial in w^3 has a critical point, i.e. $-2w^3 + 102 = 0$, or $w^3 = 51$.

Polynomials in w^3 of larger degree will have more critical points, which will be more difficult to compute and will need to be checked more carefully. (This case is extremely easy even compared to the general degree-6 case, because the quadratic in w^3 might have coefficients involving u and v instead of just rational numbers.) This is why the uvw method is best for symmetric polynomials of low degree.

1.9 Problems

Problem 1: For $a,b,c \in \mathbf{R}^+$ such that $a+b+c=1$, Prove that

$$5(a^2+b^2+c^2) \le 6(a^3+b^3+c^3)+1$$

Problem 2: Let a,b,c be three strictly positive real numbers such that $a+b+c \ge \frac{1}{a}+\frac{1}{b}+\frac{1}{c}$. Then show that

$$a+b+c \ge \frac{3}{a+b+c} + \frac{2}{abc}$$

Problem 3: Show that for whatever $a,b,c > 0$, we have

$$\sum_{cyc} \frac{a^2+b^2}{a+b} \le \frac{3(a^2+b^2+c^2)}{a+b+c}$$

Problem 4: Let $a,b,c>0$. Show that

$$\frac{1}{a+b}+\frac{1}{b+c}+\frac{1}{c+a}\le\frac{3(a+b+c)}{2(ab+bc+ca)}$$

Problem 5: Prove that for any $x,y,z\ge 0$ such that $x+y+z=3$ the following inequality holds true

$$x^2+y^2+z^2+xyz\ge 4$$

Problem 6: Let a,b,c be the lengths of the sides of ΔABC. Show that

$$\frac{a^3}{b+c-a}+\frac{b^3}{c+a-b}+\frac{c^3}{a+b-c}\ge a^2+b^2+c^2$$

Problem 7: Let $a,b,c\in\left(0,\frac{\pi}{2}\right)$, and given that $2\left(\tan(a)+\tan(b)+\tan(c)\right)=3\left(\tan(a)\cdot\tan(b)\cdot\tan(c)\right)$. Show that

$$\cos^2(a)+\cos^2(b)+\cos^2(c)\ge 1$$

Problem 8: Let x, y and z be positive numbers such that $x^3+y^3+z^3+xyz=4$ Prove that:

$$\frac{x^2+y^2}{x+y}+\frac{y^2+z^2}{y+z}+\frac{z^2+x^2}{z+x}\ge 3$$

Problem 9: Let $a,b,c\ge 0$ satisfy $ab+bc+ca=1$. Prove that

$$\frac{1+(ab)^2}{(a+b)^2}+\frac{1+(bc)^2}{(b+c)^2}+\frac{1+(ca)^2}{(c+a)^2}\ge\frac{5}{2}$$

1.10 Solutions

Solution 1

Because $p = a+b+c = 1, q = ab+bc+ca, r = abc$ implies that

$$a^3+b^3+c^3 = 3abc+a^2+b^2+c^2-ab-bc-ca = 3r+p^2-3q$$

So $5(a^2+b^2+c^2) \leq 18r+6(p^2-2q)-6q+1$

$\Leftrightarrow 18r+1-2q+1 \geq 6q$

$\Leftrightarrow 8q \leq 2+18r$

$\Leftrightarrow 4q \leq 1+9r$

$\Leftrightarrow (1-2a)(1-2b)(1-2c) \leq r$

$\Leftrightarrow (b+c-a)(c+a-b)(a+b-c) \leq r = abc.$

That is the Schurs inequality for 1.

Solution 2

Let $p = a+b+c, q = ab+bc+ca, r = abc.$

Then by the condition we have

$pr \geq q \geq \sqrt{3pr}$

so $pr \geq 3$ (1)

So, by AM-GM $a+b+c \geq 3$ (2)

Example 1.10.1. Case 1. When $r \leq 1$

Then the inequality is equivalent to $p \cdot pr \geq 3r + 2p$

Using (1) it suffices to prove that $3p \geq 3r + 2p$ or $p \geq 3r$.

But by AM-GM $p \geq \sqrt[3]{r}$ so it suffices to prove $r \leq 1$ which holds .

Example 1.10.2. Case 2. $r \geq 1$

So it suffices to prove that $p \geq \frac{3}{p+2}$

or equivalently $(p-3)(p+1) \geq 0$ which holds due to (2).

Solution 3

Let $p = a+b+c, q = ab+bc+ca, r = abc$

Rewriting the equation we get $a^2+b^2+c^2 \geq \sum_{cyc} \frac{a(b^2+c^2)}{b+c}$

$$\frac{a(b^2+c^2)}{b+c} = \frac{a((b+c)^2-2bc)}{b+c} = \frac{a(b+c)^2}{b+c} - \frac{2abc}{b+c} = ab+ac-\frac{2abc}{b+c}$$

Thus $a^2+b^2+c^2+2abc\left(\frac{1}{b+c}+\frac{1}{c+a}+\frac{1}{a+b}\right) \geq 2(ab+bc+ca)$

By Cauchy Schwartz $\frac{1}{b+c}+\frac{1}{c+a}+\frac{1}{a+b} \geq \frac{9}{2(a+b+c)}$

$\Leftrightarrow a^2+b^2+c^2+\frac{9abc}{a+b+c} \geq 2(ab+bc+ca)$

$\Leftrightarrow p^2-2q+\frac{9r}{p} \geq 2q$

$\Leftrightarrow p^3+9r \geq 4pq$ which is just Schurs.

Solution 4

If $p = a+b+c, q = ab+bc+ca, r = abc$ then

$$\frac{3(a+b+c)}{2(ab+bc+ca)} - \sum_{cyc}\frac{1}{a+b} = \frac{3p}{2q} - \frac{\sum_{cyc}(a^2+ab+bc+ca)}{(a+b)(b+c)(c+a)} = \frac{3p}{2q} - \frac{p^2+q}{pq-r}$$

$$= \frac{3p^2q-3pr-2p^2q-2q^2}{2p(pq-r)} = \frac{\frac{2q}{3}(p^2-3q)+\frac{p}{3}(pq-9r)}{2(pq-r)} \geq 0$$

Which is true in accordance with $p^2 \geq 3q$ and $pq \geq 9r$

Solution 5

We want to get all our terms to have degree 3 (since that is the highest degree of an already present term).

So

$$(x^2+y^2+z^2)\cdot\frac{x+y+z}{3}+xyz \geq 4\left(\frac{x+y+z}{3}\right)^3$$

Writing in terms of pqr we get

$$\frac{p}{3}(p^2-2q)+r \geq \frac{4p^3}{27}$$

$\Leftrightarrow 9p(p^2-2q)+27r \geq 4p^3$

$\Leftrightarrow 9p^3-18pq+26r \geq 4p^3$

$\Leftrightarrow 5p^3+27r \geq 18pq$

Now $p^3+9r \geq 4pq \rightarrow 3p^3+27r \geq 12pq$ and $p^2 \geq 3q \rightarrow 2p^3 \geq 6pq$.

Adding both we are done.

Solution 6

Let p be the semiperimeter i.e. $p=\frac{a+b+c}{2}$

Let $a=y+z, b=z+x, c=x+y$.

Thus, we have

$$\sum_{cyc}\frac{a^3}{b+c-a}=\sum_{cyc}\frac{(y+z)^3}{2x}$$

$$=\frac{\sum_{cyc} yz[y^3+z^3+3yz(p-x)]}{2r}$$

$$= \frac{(x^3+y^3+z^3)(xy+yz+zx) - xyz(x^2+y^2+z^2) + 3p(x^2y^2+y^2z^2z^2x^2) - 3r(xy+yz+zx)}{2r}$$
$$= \frac{3qr + p^3q - 3pq^2 - p^2r + 2qr + 3pq^2 - 6p^2r - 3qr}{2r}$$
$$= \frac{p^3q - 7p^2r + 2qr}{2r}$$

so, the inequality is successively equivalent to

$$\frac{p^3q - 7p^2r + 2qr}{2r} \geq \sum_{cyc} (y+z)^2$$

$$\Leftrightarrow p^3q - 7p^2r + 2pq \geq 4r(p^2 - q) \Leftrightarrow p^3q + 6qr \geq 11p^2r$$
$$\Leftrightarrow q(p^3 + 9r - 4pq) + \frac{11p}{3}(q^2 - 3pr) + \frac{q}{3}(pq - 9r) \geq 0$$

which is true using $q^2 \geq 3pr$, $pr \geq 9r$ and $p^3 + 9r \geq 4pq$.

Solution 7

Note $x = \tan a$, $y = \tan b$, $z = \tan c$.

And $a, b, c \in \left(0, \frac{\pi}{2}\right)$ so we have $x, y, z \geq 0$.

The result is $\cos^2 a = \frac{1}{1+x^2}$, $\cos^2 b = \frac{1}{1+y^2}$, $\cos^2 c = \frac{1}{1+z^2}$

So

$$\cos^2 a + \cos^2 b + \cos^2 c - 1$$
$$= \frac{1}{1+x^2} + \frac{1}{1+y^2} + \frac{1}{1+z^2} - 1$$
$$= \frac{3 + 2(x^2+y^2+z^2) + (x^2y^2+y^2z^2+z^2x^2)}{1 + (x^2+y^2+z^2) + (x^2y^2+y^2z^2+z^2x^2) + (xyz)^2)} - 1$$
$$= \frac{2 + (x^2+y^2+z^2) - (xyz)^2}{(1+x^2)(1+y^2)(1+z^2)}$$

But note that

$$2 + (x^2+y^2+z^2) - (xyz)^2$$
$$= 2 + p^2 - 2q - r^2$$

$$
\begin{aligned}
&= p^2 - 2q + \frac{3r}{p} - \frac{4}{9}p^2 \\
&= \frac{1}{9p}(5p^3 + 27r - 18pq) \\
&= \frac{2}{9}(p^2 - 3q) + \frac{1}{3p}(p^3 + 9r - 4pq) \geq 0
\end{aligned}
$$

And because $p^2 \geq 3q$ and $p^3 + 9r \geq 4pq$, of where $\cos^2 a + \cos^2 b + \cos^2 c \geq 1$ and equality holds for $x = y = z = \sqrt{2}$.

Solution 8:

Write inequality as

$$\frac{x^2+y^2}{x+y} + \frac{y^2+z^2}{y+z} + \frac{z^2+x^2}{z+x} \geqslant 3\sqrt[3]{\frac{x^3+y^3+z^3+xyz}{4}}.$$

Suppose $p = x+y+z = 3$, $q = xy+yz+zx = 3-3t^2$ $(0 \leqslant t < 1)$ and $r = abc$ inequality become

$$\frac{2p^2q - 4pr - 2q^2}{pq - r} \geqslant 3\sqrt[3]{\frac{p^3 - 3pq + 4r}{4}},$$

equivalent to

$$\frac{36 - 12r - 18t^2(t^2+1)}{9(1-t^2) - r} \geqslant 3\sqrt[3]{r + \frac{27}{4}t^2},$$

or

$$4 - \frac{6(1-t^2)(4-t^2)}{9(1-t^2) - r} \geqslant \sqrt[3]{r + \frac{27}{4}t^2}.$$

Because $r \leqslant (1+2t)(1-t)^2$, so

$$4 - \frac{6(1-t^2)(4-t^2)}{9(1-t^2) - r} \geqslant 4 - \frac{6(1-t^2)(4-t^2)}{9(1-t^2) - (1+2t)(1-t)^2} = \frac{3t^2+t+2}{t+2},$$

and

$$r + \frac{27}{4}t^2 \leqslant (1+2t)(1-t)^2 + \frac{27}{4}t^2 = 2t^3 + \frac{15}{4}t^2 + 1.$$

Therefore we need to show that

$$\frac{3t^2+t+2}{t+2} \geqslant \sqrt[3]{2t^3 + \frac{15}{4}t^2 + 1}.$$

Which is true because

$$\left(\frac{3t^2+t+2}{t+2}\right)^3 - \left(2t^3 + \frac{15}{4}t^2 + 1\right) = \frac{(4t^2+5t+6)(5t-2)^2t^2}{4(t+2)^3} \geqslant 0.$$

Solution 9:

Let $p = a+b+c, q = ab+bc+ca = 1, r = abc$

This is equivalent to

$$\frac{(ab)^2+q^2}{q(a+b)^2}+\frac{(bc)^2+q^2}{q(b+c)^2}+\frac{(ca)^2+q^2}{q(c+a)^2}-\frac{7(a-b)^2(b-c)^2(c-a)^2}{2(a+b)^2(b+c)^2(c+a)^2}-\frac{5}{2}\geq 0$$

$$\Leftrightarrow f(r) = (p^2+96q)r^2+14pq(p^2-4q)r+q^2(p^2-4q)^2 \geq 0$$

If $p^2 \geq 4q$ then we are done

If $p^2 \leq 4q$ then

$$f'(r) = \frac{2}{9}(p^2+96q)(p^3-4pq+9r)-\frac{2}{9}p(p^2+33q)(p^2-4q) \geq 0$$

Assume $p = 1$ because $(a-b)^2(b-c)^2(c-a)^2 \geq 0$.

Thus

$$r \geq \frac{1}{27}(-2p^3-2\sqrt{(p^2-3q)^3}+9pq) = \frac{1}{27}(-2-2\sqrt{(1-3q)^3}+9q)$$

Therefore

$$f'(r) = (1+96q)r^2+14q(1-4q)r+q^2(1-4q)^2 \geq f(\frac{1}{27}(-2-2\sqrt{(1-3q)^3}+9q)) \geq 0$$

Can you continue?

Takeaway from the solutions

- If you thought that there is always a solution using only this method (i.e. no other method will be needed) then you were wrong but the method does simplify the problem which makes using other methods to solve the simplified problem very easy.
- If you look back at the solutions you will see that some of the solution include other methods (such as AM-GM or Cauchy Schwarz, ex: Problem 2 or Problem 3) to prove the question. This is highly possible.

- You should try to get all the terms to the highest degree of an already present term (Problem 5). That will allow you to use the method more neatly.
- You may set up your own conventions (such as $p = \frac{a+b+c}{2}$, ex: Problem 7) but remember what you set, and change the Consequences accordingly
- These solutions may not be the most elegant and sometimes contain dirty calculation (such as expanding but these have not been included so it is suggested that the reader does so) but the power of this method is unimaginable.

Conditions

Some inequalities contain conditions like $a,b,c \geq 1$, and it is not correct to use pqr Method in this case.

Numbers a,b,c to be not less than 1

Solution: We need a,b,c are real, so p,q,r must be real and $T(p,q,r) \geq 0$ must hold. We should use non-negativity lemma for numbers $(a-1),(b-1),(c-1)$.

$$(a-1)+(b-1)+(c-1) \geq 0 \Leftrightarrow p \geq 3$$

$$(a-1)(b-1)+(b-1)(c-1)+(c-1)(a-1) \geq 0 \Leftrightarrow q-2p+3 \geq 0$$
$$(a-1)(b-1)(c-1) \geq 0 \Leftrightarrow r-q+p-1 \geq 0$$

Exercise:

Find conditions on numbers p,q,r necessary and sufficient for

1. Numbers a,b,c to be side lengths of a triangle (perhaps degenerate).

2. Non-negative real numbers a,b,c to satisfy $2 \cdot min(a,b,c) \geq max(a,b,c)$

Exercises for the reader

Problem 1:

If $a,b,c > 0$ and $a,b,c \in \mathbf{R}$ then show that

$$\sum_{cyc} \frac{1}{a^2+ab+b^2} \geq \left(\frac{3}{a+b+c}\right)^2$$

Problem 2:

Let $a,b,c > 0$, with $a+b+c = 1$. Show that

$$\frac{1}{a+b} + \frac{1}{b+c} + \frac{1}{c+a} + 3(ab+bc+ca) \geq \frac{11}{2}$$

Problem 3:

Consider a,b,c three strictly positive real numbers. Show that

$$\sum_{cyc} \frac{b+c}{a} \geq 3 + \frac{(a^2+b^2+c^2)(ab+bc+ca)}{abc(a+b+c)}$$

Problem 4:

If $a,b,c > 0$ and $a,b,c \in \mathbf{R}$ with $a+b+c = 1$. Show that

$$a^3+b^3+c^3 \geq \frac{a^2+b^2+c^2}{3}$$

Problem 5:

Let $x, y, z > 0$ and $x, y, z \in \mathbf{R}$ with $x + y + z = 3$. Show that

$$2(x^3 + y^3 + z^3) \geq x^2 + y^2 + z^2 + 3$$

Problem 6:

Let $x, y, z \in (0, \infty)$ such that $x^2 + y^2 + z^2 + 1 = 2(xy + yz + zx)$. Show that

$$9xyz \geq x + y + z$$

Problem 7:

Let $x, y, z \geq 0$ be such that $xy + yz + zx = 3$. Show that

$$4xyz(x + y + z) - 3xyz \leq 9$$

Problem 8:

Let $a, b, c \in \left(0, \frac{\pi}{2}\right)$, and given that $2(\tan(a) + \tan(b) + \tan(c)) = 3(\tan(a) \cdot \tan(b) \cdot \tan(c))$. Find the minimum value of the expression

$$\frac{1}{sin^2 a} + \frac{1}{sin^2 b} + \frac{1}{sin^2 c}$$

Problem 9:

Let $x, y, z > 0$ and $x, y, z \in \mathbf{R}$ with the property that $xy + yz + zx = 3$. Show that

$$3xyz(x + y + z) - 2xyz \leq 7$$

Problem 10:

There exist $a,b,c>0$ such that $abc=1$. Then prove the inequality that

$$\sum_{cyc}\frac{a+b}{c}\geq 2\sum_{cyc}(a+\frac{1}{a}-1)$$

Problem 11:

If $a,b,c\in\mathbf{R}^+$ with $a+b+c=1$. Show that

$$(a^2+b^2)(b^2+c^2)(c^2+a^2)\geq 8(a^2b^2+b^2c^2+c^2a^2)^2$$

Problem 12:

Let $a,b,c>0$ and $a,b,c\in\mathbf{R}$ such that $a+b+c=3$. Show that

$$abc+\frac{12}{ab+bc+ca}\geq 5$$

Problem 13:

When $a,b,c>0$ and $ab+bc+ca=3$ prove that

$$3+\frac{1}{2}\sum_{cyc}(a-b)^2\geq\sum_{cyc}\frac{a+b^2c^2}{b+c}$$

Problem 14:

When $a,b,c>0$ and $a+b+c=3$ find the minimum of

$$(3+2a^2)(3+2b^2)(3+2c^2)$$

Problem 15:

Let a,b,c be positive real numbers such that $a^2+b^2+c^2=3$. Prove that

$$\sum_{cyc}\frac{a^2+3b^2}{a+3b}\geq 3$$

Problems from Contests

Iran TST 1996

For a,b,c positive real numbers prove that

$$(ab+bc+ca)(\frac{1}{(a+b)^2}+\frac{1}{(b+c)^2}+\frac{1}{(c+a)^2})\geq\frac{9}{4}$$

Solution

The desired inequality can be written as

$$\frac{4qpr+q(p^4-2p^2q+q^2)}{(pq-r)^2}\geq\frac{9}{4}$$

Fix p and q. Since $r\leq pq$, it follows that left hand side attains its minimal value when r attains either maximal or minimal value. If $a=0$, then the desired inequality can be written as

$$\frac{(b-c)^2(b^2+bc+c^2)}{2bc(b+c)^2}\geq 0$$

Equality is attained when $a=0,b=c$.
If $a=b$, then the desired inequality can be written as $t(t-1)^2\geq 0$, where $t=\frac{c}{b}$. Equality is attained when $a=b=c$.

China Western Mathematical Olympiad 2006

Suppose that a, b, c are positive real numbers, prove that

$$1<\frac{a}{\sqrt{a^2+b^2}}+\frac{b}{\sqrt{b^2+c^2}}+\frac{c}{\sqrt{c^2+a^2}}\leq\frac{3\sqrt{2}}{2}$$

Solution

Rewrite the original inequality as follows:

$$1 \leq \frac{1}{\sqrt{1+\frac{b^2}{a^2}}} + \frac{1}{\sqrt{1+\frac{c^2}{b^2}}} + \frac{1}{\sqrt{1+\frac{a^2}{c^2}}} \leq \frac{3\sqrt{2}}{2}$$

By change of variable let $x = \sqrt{1+\frac{b^2}{a^2}}, y = \sqrt{1+\frac{c^2}{b^2}}, z = \sqrt{1+\frac{a^2}{c^2}}$ The inequality then becomes

$$1 \leq \frac{1}{x} + \frac{1}{y} + \frac{1}{z} \leq \frac{3\sqrt{2}}{2}$$

The conditions can be rewritten as:

$$x,y,z \geq 1 \Leftrightarrow p-3 \geq 0, q-2p+3 \geq 0, r-q+p-1 \geq 0$$

India TST 2017

Let a,b,c be distinct positive real numbers with $abc = 1$. Prove that

$$\sum_{\text{cyc}} \frac{a^6}{(a-b)(a-c)} > 15$$

Solution

Due to the conditions $a+b+c > 3$, and using standard pqr notation we can easily find that

$$S = \sum_{cyc} \frac{a^6}{(a-b)(a-c)} = p^2(p^2-3q) + 2pr + q^2,$$

Then $S = p^2(p^2-3q)+2pr+q^2 > 2pr+q^2 = 2abc(a+b+c)+(ab+bc+ca)^2 > 6+9 = 15.$

USAMO 2001

Let $a,b,c \geq 0$ and satisfy

$$a^2+b^2+c^2+abc = 4.$$

Show that

$$0 \leq ab+bc+ca-abc \leq 2.$$

Solution

Put $x = p = a+b+c, y = p^2 - 2q = a^2+b^2+c^2, z = r = abc$.

a,b,c are real, if $T(x,y,z) = T(x, \frac{x^2-y}{2}, z) \geq 0$. It is also clear that if $T(x,y,z) = 0$, then two variables from a,b,c are equal.The non-negativity theorem is rewritten as follows:

$$p \geq 0 \Leftrightarrow x \geq 0, q \geq 0 \Leftrightarrow x^2 - y \geq 0, r \geq 0 \Leftrightarrow z \geq 0$$

The condition $a^2+b^2+c^2+abc = 4$ is rewritten as $y+z=4$. Inequality $ab+bc+ca-abc \geq 2$ can be rewritten as $x^2 - y - 2z \leq 4$.Let us fix y and z. It is enough to check the inequality for the maximum x (max - a singular x exists since $x \leq 3(a^2+b^2+c^2) \leq 12)$.

In the respective, the triple a,b,c two variables are equal. Substituting $a = b$ into the original inequality, we obtain the correct inequality. Equality cases: $a = b = c = 1$ and $a = b = \sqrt{2}, c = 0$.

JBMO SL 2019

Show that for any positive real numbers a,b,c such that $a+b+c = ab+bc+ca$, the following inequality holds

$$3 + \sqrt[3]{\frac{a^3+1}{2}} + \sqrt[3]{\frac{b^3+1}{2}} + \sqrt[3]{\frac{c^3+1}{2}} \leq 2(a+b+c)$$

Solution

Since $f(x) = \sqrt[3]{x}$ is concave for $x \geq 0$, by Jensens Inequality we have

$$\sum_{cyc} \sqrt[3]{\frac{a^3+1}{2}} \leq \sqrt[3]{\frac{a^3+b^3+c^3+3}{6}}$$

So it is enough to prove that

$$\sqrt[3]{\frac{a^3+b^3+c^3+3}{6}} \leq \frac{2(a+b+c)-3}{3}$$

Now writing in pqr notation we get $p = a+b+c = ab+bc+ca = q, r = abc$ Note that

$p^2 \geq 3q = 3p \implies p \geq 3$.
Thus it is enough to show that

$$\frac{p^3 - 3p^2 + 3r + 3}{6} \leq \frac{(2p-3)^3}{27}$$

After expanding, this is equivalent to

$$7p^3 - 45p^2 + 108p - 27r - 81 \geq 0$$

But we know that $p^3 \geq 27r$, so it is enough to prove that

$$6p^3 - 45p^2 + 108p - 81 = 3(p-3)^2(2p-3) \geq 0$$

which is evident as $p \geq 3$.

Problems from Contests for the Reader

Poland 1991

Let a,b,c be positive reals with $a^2+b^2+c^2=2$.Prove the inequality

$$a+b+c \leq 2+abc$$

Stronger: Let a, b, c be reals such that $a^2+b^2+c^2=1$.Prove that

$$a+b+c-2abc \leq \sqrt{2-5a^2b^2c^2}.$$

Romania JBMO TST 2009

Let $a,b,c>0$ be real numbers with the sum equal to 3. Show that:

$$\frac{a+3}{3a+bc}+\frac{b+3}{3b+ca}+\frac{c+3}{3c+ab} \geq 3$$

APMO 2004

Prove that the inequality

$$\left(a^2+2\right)\left(b^2+2\right)\left(c^2+2\right) \geq 9\,(ab+bc+ca)$$

holds for all positive reals a, b, c.

IMO SL 2011

Let a,b and c be positive real numbers satisfying $\min(a+b,b+c,c+a)>\sqrt{2}$ and $a^2+b^2+c^2=3$. Prove that

$$\frac{a}{(b+c-a)^2}+\frac{b}{(c+a-b)^2}+\frac{c}{(a+b-c)^2} \geq \frac{3}{(abc)^2}.$$

Korea 2012

a, b, c are positive numbers such that $a^2 + b^2 + c^2 = 2abc + 1$. Find the maximum value of

$$(a - 2bc)(b - 2ca)(c - 2ab)$$

Russia 2015

Positive real numbers a, b, c satisfy

$$2a^3b + 2b^3c + 2c^3a = a^2b^2 + b^2c^2 + c^2a^2.$$

Prove that

$$2ab(a-b)^2 + 2bc(b-c)^2 + 2ca(c-a)^2 \geq (ab + bc + ca)^2.$$

1.11 Alternative forms of Schurs Inequality

Assuming $a,b,c \geq 0$ we have the following results:

$$a^t(a-b)(a-c)+b^t(b-c)(b-a)+c^t(c-a)(c-b) \geq 0$$

Result 1

$t=0$ we have $a^2+b^2+c^2 \geq ab+bc+ca$

Result 2

$t=1$ expanded we get $a^3+b^3+c^3+3abc \geq a^2b+b^2c+c^2a+ab^2+bc^2+ca^2$

Result 3

$t=1$ expanded we get $abc \geq (-a+b+c)(a-b+c)(a+b-c)$

Result 4

$t=1$ expanded we get

$$(a+b+c)^2+\frac{9abc}{a+b+c} \geq 4(ab+bc+ca)$$

Result 5

$t=2$ expanded we get

$$a^4+b^4+c^4+abc(a+b+c) \geq a^3b+b^3c+c^3a+ab^3+bc^3+ca^3$$

Result 6

Schurs inequality of 3^{rd} degree is equivalent with

$$\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}+\frac{4abc}{(a+b)(b+c)(c+a)}\geq 2$$

Result 7

Schurs inequality of 4^{th} degree can be rewritten into

$$(a+b+c)(a^3+b^3+c^3+3abc)\geq 2(a^2+b^2+c^2)(ab+bc+ca)$$

Result 8

For $a,b,c>0$ and $abc=1$ we have

$$(a-1)\left(\frac{1}{b}-1\right)+(b-1)\left(\frac{1}{c}-1\right)+(c-1)\left(\frac{1}{a}-1\right)\geq 0$$

(Let $a=\frac{x}{y}$ etc, then this is Schurs inequality of 0^{th} degree).

Result 9

3^{rd} degree

$$(a^2+b^2+c^2)(a+b+c)+9abc\geq 2(a+b+c)(ab+bc+ca)$$

Result 10

Weaker form of Result 4

$$a^2+b^2+c^2+2abc+1\geq 2(ab+bc+ca)$$

Result 11

Stronger than 3^{rd} degree, equivalent with 4^{th} degree

$$a^3+b^3+c^3+3abc \geq \sum bc(b+c)+\frac{bc(b-c)^2+ca(c-a)^2+ab(a-b)^2}{a+b+c}$$

Result 12

4^{th} degree

$$a^2+b^2+c^2+\frac{6abc(a+b+c)}{a^2+b^2+c^2+ab+bc+ca} \geq 2(ab+bc+ca)$$

Result 13

$$\left(\frac{a}{b+c}\right)^2+\left(\frac{b}{c+a}\right)^2+\left(\frac{c}{a+b}\right)^2+\frac{10abc}{(a+b)(b+c)(c+a)} \geq 2$$

Result 14

Stronger than Schur of 3^{rd} degree, but weaker than 5^{th} degree

$$\frac{a^2+b^2+c^2}{ab+bc+ca}+\frac{8abc}{(a+b)(b+c)(c+a)} \geq 2$$

Result 15

Schur of 5^{th} degree

$$a^2+b^2+c^2+\frac{6abc}{a+b+c}+\frac{abc(a+b+c)}{a^2+b^2+c^2} \geq 2(ab+bc+ca)$$

Eulers inequality

$R \geq 2r$ where R,and r are the circumradius and inradius respectively, this expressed in sides a, b, c we get

$$\frac{abc}{\sqrt{(a+b+c)(-a+b+c)(a-b+c)(a+b-c)}} \geq \frac{\sqrt{(-a+b+c)(a-b+c)(a+b-c)}}{a+b+c}$$

which if rewritten is Result 3.

Gerretsens Inequality

If s, r, R denotes the semiperimeter, inradius and circumradius of a triangle, then

$$16Rr - 5r^2 \leq s^2 \leq 4R^2 + 4Rr + r^2$$

Shortcuts

One important thing to note is that all of the summations below are cyclic.

$$\sum x = p$$

$$\sum x^2 = p^2 - 2q$$
$$\sum x^3 = p^3 - 3pq + 3r$$
$$\sum x^4 = p^4 - 4p^2q + 4pr + 2q^2$$
$$\sum x^5 = p^5 - 5p^3q + 5p^2r + 5pq^2 + 5qr$$
$$\sum x^6 = p^6 - 6p^4q + 6p^3r + 9p^2q^2 - 2q^3 - 12pqr + 3r^2$$

$$\sum xy = q$$
$$\sum (xy)^2 = q^2 - 2pr$$
$$\sum (xy)^3 = q^3 - 3pqr + 3r^2$$
$$\sum (xy)^4 = q^4 - 4pq^2r + 2p^2r^2 + 4qr^2$$
$$\sum (xy)^5 = q^5 - 5pq^3r + 5p^2qr^2 + 5q^2r^2 - 5pr^3$$

$$\sum (x^2y + xy^2) = pq - 3r$$
$$\sum (x^3y + xy^3) = p^2q - 2q^2 - pr$$
$$\sum (x^4y + xy^4) = p^3q - 3pq^2 - p^2r + 5qr$$
$$\sum (x^5y + xy^5) = p^4q - p^3r - 4p^2q^2 + 7pqr + 2q^3 - 3r^2$$

$$\sum x^2yz = pr$$
$$\sum x^3yz = p^2r - 2qr$$
$$\sum x^4yz = p^3r - 3pqr + 3r^2$$
$$\sum x^2y^2z = q^2r - 2pr^2$$

$$\sum (x^3y^2z + x^3yz^2) = pqr - 3r^2$$
$$\sum (x^3y^2 + x^2y^3) = pq^2 - 2p^2r - qr$$

$$\sum(x^4y^2+x^2y^4)=p^2q^2-2q^3-2p^3r+4pqr-3r^2$$
$$\sum(x^4y^3+x^3y^4)=pq^3-3p^2qr+5pr^2-q^2r$$

$$\sum(x+y)(y+z)=p^2+q$$
$$\sum(1+x)(1+y)=2p+q+3$$
$$\sum(1+x)^2(1+y)^2=2p^2+2pq-2pr+q^2+4q-6r+3$$
$$\sum(x+y)^2(y+z)^2=p^4-p^2q+q^2-4pr$$
$$\sum(x^2+xy+y^2)(y^2+yz+z^2)=p^4-3p^2q+3q^2$$

$$\prod(x+y)=pq-r$$
$$\prod(1+x)=1+p+q+r$$
$$\prod(1+x^2)=p^2+q^2+r^2-2pr-2q+1$$
$$\prod(1+x^3)=p^3+q^3+r^3-3pqr-3pq-3r^2+3r+1$$
$$\prod(x^2+xy+y^2)=p^2q^2-3q^3-p^3r$$
$$\prod(x^2y+y^2z+z^2x)(xy^2+yz^2+zx^2)=p^3r+9r^2-6pqr+q^3$$
$$(x^3y+y^3z+z^3x)(xy^3+yz^3+zx^3)=p^5r-5p^3qr+pq^2r+7p^2r^2+q^4$$

$$\prod(x-y)^2=-4p^3r+p^2q^2+18pqr-4q^3-27r^2$$
$$\sum(x^2y-x^2z)=-(x-y)(y-z)(z-x)$$
$$\sum(x^3y-x^3z)=-p(x-y)(y-z)(z-x)$$
$$\sum(x^4y-x^4z)=-(p^2+q)(x-y)(y-z)(z-x)$$
$$\sum(x^5y-x^5z)=-(p^3-2pq+r)(x-y)(y-z)(z-x)$$
$$\sum(x^3y^2-x^2z^3)=-q(x-y)(y-z)(z-x)$$
$$\sum(x^4y^2-x^2z^4)=-(pq-r)(x-y)(y-z)(z-x)$$

1.12 References

Here is a list of everything that I went through and you should do too.

[1] Nguyen Anh Cuong, ABC concreteness method, 2005.

[2] Michael Rozenberg, *uvw* method, 3-4 (59-60), 2011.

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[6] Andi Gabriel BROJBEANU, Procedeu de demonstrare a unor inegalitati bazat pe inegalitatea lui Schur.

[7] Brilliant.org The *uvw* Method